On Noncommutative Solitons

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Abstract

We consider 2+1-dimensional classical noncommutative scalar field theory. The general ansatz for a radially symmetric solution is obtained. Some exact solutions are presented. Their possible physical meaning is discussed. The case of the finite θ is discussed qualitatively and illustrated by some numerical results.

1   Introduction

Noncommutative field theories became an object of intense study during the last year due to their close connection to string theory. Namely, in the large B-field limit the associative string field algebra $\mathcal{A}$ factors into an algebra that acts on the string center of mass and an algebra that describes all other degrees of freedom[1,2,3]. The second algebra seems to be terribly complicated while the first one leads to the effective noncommutative theory. Specifically, description of the scalar field theory of tachyons on the world-volume of unstable Dp-branes in superstring theory is a field of employment for noncommutative theory[4].

Of especial interest are solutions of the equation $\phi \star \phi = \phi$, i.e. projection operators. Here $\star$ denotes the nonlocal (Moyal) star product that becomes

$$f \star g(x) = e^{\frac{i}{2}\epsilon^{\mu\nu}\partial'_\mu \partial''_\nu} f(x')g(x'')|_{x'=x''=x} \tag{1}$$

when written in rescaled coordinates. If one goes to the momentum space,

$$\tilde{f}(k) = \int \frac{d^2x}{2\pi} f(x)e^{-ikx}, \tag{2}$$

then the $\star$-product takes the form

$$\widetilde{f \star g}(p) = \int \frac{d^2k}{2\pi} \tilde{f}(k)\tilde{g}(p-k)e^{-\frac{i}{2}\epsilon^{\mu\nu}k_\mu p_\nu}. \tag{3}$$

After rescaling the coordinates the (static) energy functional is given by

$$E[\phi] = \frac{1}{g^2} \int d^2x (\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \theta V(\phi)). \tag{4}$$

Here $\frac{\partial}{\partial\phi}V(\phi)|_{\phi=0} = 0$ is implied, θ is the noncommutativity parameter[5]. All the fields are multiplied using the $\star$-product.

The trick used to solve the equations of motion is to formally identify the noncommutative quantum space on which the ϕ-field lives with the classical phase space of a single particle and to recast the problem into the operator formalism using the Weyl-Moyal (WM) correspondence. The latter is the correspondence between the multiplication (composition) of operators acting on a single-particle Hilbert space and the $\star$-product of functions on the corresponding phase space. It is defined by the commutativity of the following diagram:

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$$
\begin{array}{ccc}
\hat{f}, \hat{g} & \longrightarrow & \hat{f}\hat{g} \\
\Big\downarrow \Omega^{-1} & & \Big\downarrow \Omega^{-1} \\
f, g & \longrightarrow & f \star g
\end{array}
$$

Ω denotes a map (more precisely, one-one correspondence) from the algebra of the functions to the algebra of corresponding operators using symmetric (Weyl) ordering prescription:

$$
\Omega : f(p,q) \rightarrow \hat{f}(\hat{p},\hat{q}) = \int \frac{d^2 k}{2\pi} \tilde{f}(k_p, k_q) e^{i(k_p \hat{p} + k_q \hat{q})}, \tag{5}
$$

$\tilde{f}(k)$ being defined by (2). Thus the $\star$-product [1] provides such a deformation of the algebra of functions on the phase space (which is restricted to be a symplectic manifold with a constant Poisson bi-vector $\theta^{\mu\nu}$) that Ω becomes an isomorphism[6]. The following relation also holds:

$$
\int \frac{d^2 x}{2\pi} \phi(x) = Tr_{\mathcal{H}} \hat{\phi}. \tag{6}
$$

If the Hilbert space $\mathcal{H}$ is the space of wave functions in coordinate representation, there exists the explicit formula to map a kernel of a Hilbert-Schmidt operator acting on $\mathcal{H}$ into the corresponding function

$$
f(p,q) = \int\int dx dy K(x,y) e^{-ip(x-y)} \delta(q - \frac{x+y}{2}) \tag{7}
$$

such that $\langle x \mid \hat{f} \mid y \rangle = K(x,y)$; Weyl ordering has been used[7]. The inverse formula reads

$$
K(x,y) = \int\int \frac{dp dq}{2\pi} f(p,q) e^{ip(x-y)} \delta(q - \frac{x+y}{2}) \tag{8}
$$

and becomes a kind of Fourier transform after integration over q (in order to obtain these two formulae one reorders $\hat{p}, \hat{q}$ in the exponent in the rhs of (5) using the Baker-Campbell-Hausdorff formula and calculates the matrix elements directly).

The following treatment beyond the WM correspondence also exists: one can avoid explicit mentioning the WM correspondence at all and omit the Hilbert space of wavefunctions by using the integral transform (7) immediately, then the $\star$-product of two functions $f \star g$ corresponds to the convolution of their kernels (i.e. integral transforms(8)): [2]

$$
K_{f \star g}(x,y) = \int d\xi K_f(x,\xi) K_g(\xi,y). \tag{10}
$$

In the large noncommutativity limit one can neglect the kinetic term and obtain the following equation of motion: $\frac{\partial V}{\partial \phi} = 0$, where $V(\phi)$ is considered a polynomial or an analytic function (note that the constant term in the lhs is absent). Projection operators are the basic thing to solve such an equation. A localized solution will be referred to as a soliton.

In the next section the general form of any solution radially symmetric w.r.t. spatial coordinates is derived. The third section deals with some distribution-valued solutions and related topics. The fourth one is reasoning, mainly perturbative and numeric, how does the kinetic term affect the physics at finite θ.

2 General radially symmetric solution

In[5] radially symmetric projection operators diagonal in the holomorphic representation and corresponding functions on the phase space have been worked out. The reason for the present

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[1]Here the same as in (1),(3). In our considerations $\hbar = 1$.

[2]To verify this one writes the δ-function in (7) in the momentum representation

$$
\delta(q - \frac{x+y}{2}) = \int \frac{d\xi}{2\pi} e^{i\xi(q - \frac{x+y}{2})} \tag{9}
$$

or, equivalently, uses (3).

examination is that in general adding some non-radially symmetric functions ($\mid m\rangle\langle n\mid$'s in the operator formalism, where $m \neq n$) or composing a series from them one can in principle obtain a radially symmetric result (for example, $p^2 + q^2 = r^2$). In this section we are to construct the general ansatz for any radially symmetric solution. In order to do this we examine how ϕ changes under an infinitesimal coordinate rotation $\delta q = \epsilon p$, $\delta p = -\epsilon q$:

$$\delta\phi(p,q) = \epsilon(p\frac{\partial\phi}{\partial q} - q\frac{\partial\phi}{\partial p}) + o(\epsilon). \tag{11}$$

(7) can be rewritten as

$$\phi(p,q) = 2\int\limits_{-\infty}^{+\infty} d\xi K(q-\xi, q+\xi)e^{2ip\xi}. \tag{12}$$

After use of (11) one calculates (up to the first order in ϵ)

$$0 = \int\limits_{-\infty}^{+\infty} d\xi(p(\partial_x + \partial_y)K(x,y) - 2iq\xi K(x,y))|_{x=q-\xi, y=q+\xi}e^{2iP\xi}. \tag{13}$$

Being multiplied by $e^{-2ip\xi'}$ and integrated w.r.t. p, it becomes

$$0 = \int\limits_{-\infty}^{+\infty} d\xi(\frac{i}{2}(\partial_x + \partial_y)K(x,y)\partial_{\xi'} - 2iq\xi K(x,y))|_{x=q-\xi, y=q+\xi}\delta(\xi-\xi'), \tag{14}$$

what simplifies to [3]

$$(-\frac{1}{2}\partial_{xx} + \frac{1}{2}x^2)K(x,y) = (-\frac{1}{2}\partial_{yy} + \frac{1}{2}y^2)K(x,y). \tag{16}$$

Separating variables $K(x,y) = \Psi_x(x)\Psi_y(y)$ one finds

$$(-\frac{1}{2}\partial_{xx} + \frac{1}{2}x^2)\Psi_x(x) = E\Psi_x(x),$$
$$(-\frac{1}{2}\partial_{yy} + \frac{1}{2}y^2)\Psi_y(y) = E\Psi_y(y). \tag{17}$$

We are looking for such kernels that convolution of K with K exists (i.e. the integral in the rhs of (10) converges). This means that Ψ_x (and Ψ_y) vanishes sufficiently rapidly when $x \to \infty$ (correspondingly $y \to \infty$). Similar asymptotic conditions follow from the finiteness of energy. [4] There exist some solutions for (17) only when $E = (n + \frac{1}{2})$ and those are the wave functions of the 1d harmonic oscillator, i.e. the basis of the holomorphic representation discussed in[5]. Thus any radially symmetric solution can be represented as $\sum\limits_{n=0}^{\infty} C_n\phi_n$, where C_n are some coefficients and ϕ_n are solutions built in[5].

3 On some generalized solutions

Explicit construction. Now we proceed to apply these considerations to the case of the SUSY even ϕ^4 potential

$$V(\phi) = -\frac{1}{2}\phi^2 + \frac{1}{4}\phi^4 \tag{18}$$

The "improper" sign to the quadratic term can describe tachyons as their squared mass is negative. On the other hand one can make a constant shift and substitute $\phi \to \phi + const$

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[3]This equation is Fourier self-dual and remains unchanged under the transform

$$K(x,y) = \int \frac{d^2k}{2\pi}\tilde{K}(k)e^{i(k_x x - k_y y)}. \tag{15}$$

[4]Thus the WM correspondence is valid.

thus expanding the potential near the minimum. In the large noncommutativity limit the equation of motion becomes

$$-\phi + \phi^3 = 0. \tag{19}$$

It has been pointed out[8] that $\phi(x) = \pi\delta^2(x)$ satisfies such an equation and[5,8]

$$\pi\delta^2(x) \star \pi\delta^2(x) = 1. \tag{20}$$

Speaking commonly, how should one understand such a relation? Here we have $\pi\delta^2(x)$, a singular distribution with a compact measure 0 support which can be dealt with as a weak limit of a function $\Phi(x, \epsilon)$: for instance,

$$\Phi(x, \epsilon) = \frac{1}{2\pi\epsilon} e^{-\frac{x^2}{2\epsilon}}, \epsilon \to +0 \tag{21}$$

(momentum representation with a regularization factor equal to $e^{-\frac{\epsilon k^2}{2}}$). Then carrying out the $\star$-multiplication in the momentum space, one finds that divergences (products of singular distributions and their derivatives) cancel each other (what may look like a miracle and happens due to the nonlocality of the $\star$-product). Considering a function like $\phi(x) = \pi p(x)\delta(x)$, where $p(x)$ is some polynomial, one finds that the Fourier transform mapping into the momentum space (and, consequently, the $\star$-multiplication) does not distinguish between "$p(x)$" and "$p(0)$". This implies that any function

$$\phi(x) = \pi p(x)\delta^2(x) \tag{22}$$

such that $p(x)$ is a polynomial and

$$p(0) = 1 \tag{23}$$

solves (19). At first glance this result seems both evident and trouble-looking, so a little discussion on nonlocal product of distributions seems to be necessary.

First we note that

$$\partial_x^k(f(x)\delta(x)) = f(0)\partial_x^k\delta(x) \tag{24}$$

in a weak sense (as a distribution), where f is a regular function. As an evidence for this one can integrate both parts of the above equation with an arbitrary test function and perform integration by parts k times. This observation is to prove the dependence only on $f(0)$.

Another question is to prove the convergence and to calculate the value of the integral in the momentum space. No uniform convergence w.r.t. the regularization parameter (like ϵ in (21)) seems to be valid, but one imposes another regularization in the momentum space and successfully calculates the result of the $\star$-multiplication. This approach is the basic one in quantum field theory and the distinction in our case is that no renormalization is needed. That is why we say "the divergences cancel each other". So being rigorous, the result in the momentum space also converges as a distribution.

The third thing is the series

$$\pi\delta^2(x) = \sum_{n=0}^{\infty} (-1)^n \phi_n(x) \tag{25}$$

converges only as a distribution. For example, replacing $(-1)^n$ with $(-\epsilon)^n$, one gets a suitable regularization. In particular the vacuum energy becomes finite and equal to

$$E(\epsilon) = -\frac{1 + 2\epsilon^2}{4(1 - \epsilon^4)}, \tag{26}$$

but $E \to -\infty$ as $\epsilon \to 1 - 0$. Evidence that the configuration $C_n = (-1)^n$ is situated at the boundary point of the strong convergence range enables one to put forward the equivalent mathematical approach to the polynomial $p(x)$ as a factor changing the form of the regularization.

The above considerations can also be applied to power series with reasonable asymptotic behaviour of the coefficients.

It is interesting to see what is the corresponding operator in the Hilbert space. From (8) one easily calculates $K(x, y) = \delta(x + y)$, i.e. this is the spatial reflection $\hat{P}$ in imaginary particle's configuration space. That is the obvious reason why it is no projector. Corresponding coefficients C_n are simply the matrix elements of P in the holomorphic representation: $C_n = (-1)^n$ as the harmonic oscillator wavefunctions have definite parity. For the unity operator, of course, $C_n = 1$ and we can construct a projector onto even states $\frac{1}{2}(\hat{I} + \hat{P})$, $\phi = \frac{1}{2}(1 + \pi p(x)\delta^2(x))$.

Classical stability. Let us examine whether our solution $C_n = (-1)^n$, $\theta = \infty$ is stable. For this purpose we expand the energy functional near this configuration up to the second order terms and check if the quadratic form to appear is positively defined. No radial symmetry of the infinitesimal field change $\delta\phi$ is implied. The second variation proves to be (in operator formalism) [5]

$$\delta E[\phi] = \frac{2\pi\theta}{g^2}Tr_{\mathcal{H}}(-\frac{1}{2}(\widehat{\delta\phi})^2 + \frac{1}{4}(4\hat{\phi}^2\widehat{\delta\phi}^2 + 2(\hat{\phi}\widehat{\delta\phi})^2)), \tag{27}$$

what by use of $\hat{\phi}^2 = \hat{I}$ reduces to

$$\delta E[\phi] = \frac{\pi\theta}{g^2}Tr_{\mathcal{H}}(\widehat{\delta\phi}^2 + (\hat{\phi}\widehat{\delta\phi})^2) > 0. \tag{28}$$

Thus the solution is stable. The same considerations apply to $\phi = 1$, $C_n = 1$. It is clear that any solution of (19) to satisfy

$$\phi \star \phi = 1 \tag{29}$$

is classically stable while the solution for which exists n such that $C_n = 0$ is unstable, for instance, to the radial fluctuations of the n-th mode because the quartic term gives no rise to the quadratic part of expansion now.

Physical interpretation. First, "localization" of solution is understood comparatively as the energy of the δ-like solution is infinite, but it is perfectly localized spatially.

Coefficients of the polynomial $p(x)$ are unlikely to be important on the classical level and seem to be some internal degrees of freedom. The question concerns their meaning.

In general there exist some different treatments with noncommutative solitons.

1. Such a localized 2d configuration can be considered to be a D23-brane within D25 brane[8,9]. The correct brane tension is recovered from the action calculated on the noncommutative soliton. $P(x)$ can describe excitations living on the world volume of this D23-brane.

2. In[8] it was proposed that noncommutative solitons are the classical vacua for the compact scalar field theory. In our case this statement is supported by the fact that

$$E[\phi]|_{\phi=\pi\delta^2(x)} = E[\phi]|_{\phi=1} = -\infty, \tag{30}$$

while the energy density remains constant and equal to

$$\varrho_E = -\frac{\theta}{4g^2}. \tag{31}$$

The energy density does not depend on the sign of C_n if $C_n = \pm 1$ at all as ϕ_n are $\star$-orthogonal. Then such classical vacua can be parametrized by the infinite number of coefficients $C_n = \pm 1$. The coefficients of the polynomial $p(x)$ are some excitations. Similar considerations concerning the energy density partially apply to the next item.

Following these ideas, it is interesting to exhibit an example of an instanton interpolating between such vacua. To do this we first consider the scalar self-interacting field theory in 2+1-dimensional Minkowski space-time with the $(+--)$ signature and commutative timelike direction. The action functional is [6]

$$S[\phi] = \frac{1}{g^2}\int d^3x(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)), \tag{32}$$

from what one reads the energy functional (4) and (now in rescaled noncommuting space coordinates) the non-static equation of motion

$$\frac{\partial^2\phi}{\partial t^2} = \frac{1}{\theta}\triangle\phi - \frac{\partial V}{\partial\phi}. \tag{33}$$

When $\theta = \infty$ the first term in the rhs can be neglected. In the euclidean time (or, equivalently, inverted potential)

$$\frac{\partial^2\phi}{\partial t^2} = \frac{\partial V}{\partial\phi}. \tag{34}$$

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[5]We have performed some cyclic permutations under the trace sign. Field variations are considered to be small.

[6]No rescaling has been implied yet.

Now it is easy to construct an example of an instanton tunneling between two noncommutative vacua. Should one restrict the solution to be diagonal at any time moment $-\infty < t < \infty$, the equation of motion becomes the infinite set of decoupled equations

$$\frac{d^2}{dt^2}C_n = C_n - C_n^3, \tag{35}$$

each of them describes the well known quantum mechanical double well kink if the coefficient C_n is tunneling from -1 to 1, antikink $(1 \to -1)$ or a constant $(1 \to 1$ or $-1 \to -1$, no tunneling).

This construction can be applied to the dynamics of radially symmetric solitons and the equations of motion are written in the Minkowski space-time. Recently some new results concerning noncommutative soliton scattering[10] were obtained using the adiabatic approximation. [7] The typical feature is that the equations (35) are identical for all modes. If all the initial conditions are the same (up to a sign), the solution is $\phi(r, t) = C(t)\Phi(r)$, i.e. the time dynamics reduces to the scaling factor in front of the soliton. As $V(\phi) \to \infty$ when $\phi \to \infty$, all the coefficients C_n remain finite.

If small excitations mentioned above are radially symmetric, they can be described as small oscillations of C_n near the minimum of potential.

3. Some vortex-like solutions in noncommutative abelian Higgs model (which has similar potential for the scalar field) have recently been discussed[11]. In this context (and also for the brane interpretation) the subtlety to arise is the uniform energy density while it is significantly singular for a commutative brane or a vortex. It is not completely clear for what vortex's internal degrees of freedom $p(x)$ is responsible.

4   Behaviour of solutions with finite θ

Recurrent relation.   The question still unanswered completely is how does the kinetic term affect the physics at finite θ. To examine some aspects one should write the equations of motion at finite θ. It is known[5,12] that in the operator formalism

$$\frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi \leftrightarrow [\hat{a}, \hat{\phi}][\hat{\phi}, \hat{a}^\dagger], \tag{36}$$

where $\hat{a}^\dagger$, $\hat{a}$ are creation and annihilation operators defined in Appendix. Performing variation in (4) w.r.t. ϕ, one obtains

$$[\hat{a}^\dagger, [\hat{a}, \hat{\phi}]] + [[\hat{\phi}, \hat{a}^\dagger], \hat{a}] = -\theta\frac{\partial V}{\partial \phi}(\hat{\phi}). \tag{37}$$

By use of Jacobi identity and commutation relations for creation and annihilation operators lhs yields

$$[\hat{a}^\dagger, [\hat{a}, \hat{\phi}]] = -\frac{1}{2}\theta\frac{\partial V}{\partial \phi}(\phi). \tag{38}$$

With the radially symmetric ansatz the equation of motion simplifies to the recurrent relation [8]

$$(n+1)C_{n+1} - (2n+1)C_n + nC_{n-1} = \frac{1}{2}\theta V^{'}(C_n), n > 0,$$
$$C_1 - C_0 = \frac{1}{2}\theta V^{'}(C_0), n = 0. \tag{39}$$

Should one try to find an appropriate potential $V(\phi)$ for a solution $C_n = (-1)^n A$ to exist, it will end in no positive result as in this case (39) reduces to

$$2(-1)^{n+1}(2n+1)A = \frac{\theta}{2}V^{'}((-1)^n A). \tag{40}$$

As it has been emphasized, an equation like (39) is hard to solve, but some effects could be studied qualitatively or numerically.

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[7]The calculations below are closed for $\theta = \infty$.

[8]For derivation see Appendix.

Qualitative effects. 1.The differential equation for this scheme is

$$nC'' + C' = \frac{1}{2}\theta C(C^2 - 1). \tag{41}$$

A similar one occurs in the construction of the usual commutative Abrikosov-Nielsen-Olesen vortex and the only acceptable asymptotic for us is that the coefficients $C(n) = C_n$ vanish at the infinity. The mode expansion $\phi = \sum C_n \phi_n$ is somewhat like a discrete Fourier transform. Then the singular (or at least localized) vortex-like solution in the discrete momentum space should lead to delocalization in the field ϕ-space. Thus, the δ-like solution becomes finite-size. The effect of finite θ is imposing a regularization.

2.The energy functional is[12]

$$E[\phi] = \frac{1}{g^2} \sum_{n=0}^{\infty} ((2n+1)C_n^2 - 2(n+1)C_{n+1}C_n + \theta V(C_n)). \tag{42}$$

Analyzing the first two terms as a perturbation, i.e. simply calculating the energy functional for the solution at $\theta = \infty$ one observes that the second addend in the kinetic term causes the level splitting and breaks the 2^∞-times degeneration. Perfectly localized δ-like solution becomes the highest energy level because all the perturbative terms in (41) appear to be positive for the coefficients with alternating sign. This is the reason to understand of what importance is the δ-like radially symmetric solution. To do that one observes that the results for $\theta = \infty$ are true for any set of mutually orthogonal projectors. It is the kinetic term that causes the $U(\infty)$ symmetry breakdown. For an arbitrary operator like

$$\hat{\phi} = U^\dagger diag(\{\lambda_n\})U, \tag{43}$$

with U being unitary the energy functional is equal[12] up to the coupling constant

$$E[\{\lambda_n\}, \{U_{mn}\}] = \sum_{n=0}^{\infty} \lambda_n^2 (1 + 2\sum_{m=0}^{\infty} m|U_{mn}|^2) - 2\sum_{m,n=0}^{\infty} \lambda_m \lambda_n |A_{mn}|^2 + \theta \sum_{n=0}^{\infty} V(\lambda_n), \tag{44}$$

where

$$A_{mn} = \sum_{k=1}^{\infty} \sqrt{k} U_{kn} U_{k-1,m}^*, \tag{45}$$

If we hold $\{\lambda_n\}$ fixed, then the first order variation of E w.r.t. U is

$$\delta E = \frac{\partial E}{\partial U_{mn}}|_{U=1}\delta U_{mn} + \frac{\partial E}{\partial \bar{U}_{mn}}|_{U=1}\delta \bar{U}_{mn} =$$
$$2\sum_{n=0}^{\infty} (3n+1)(\delta U_{nn} + \delta \bar{U}_{nn}) = 0 \tag{46}$$

as in the first order

$$\delta U_{nn} + \delta \bar{U}_{nn} = 0 \tag{47}$$

for U is unitary.

Thus for $\theta = \infty$ the matrix U serves as coordinates along the vacuum valley, but at finite θ the special significance of radially symmetric solutions becomes clear.

Numeric study. Our method is different from that in[12]. It is clear that specifying C_0 defines all the other coefficients and provides an opportunity for the numeric study. In our study $\theta = 100$ and remains constant. The main goal is to see how does a finite-θ solution look like. Some graphs are presented below for $C_0 = 10^{-10}$.

The equations are strongly nonlinear, so as we start from $C_0 = 10^{-10}$ it first grows in ten orders up to $C \sim 1$ and then slowly converges to 0 (see the first graph). In the second graph we show the sum of the 100 first terms in the series for $\phi(r)$. The third one illustrates that $\phi(0)$ remains finite and thus the solution is finite-size.

Another interesting phenomenon is that a small change in C_0 can make the solution divergent. For example, there exist at least two disconnected ranges of convergence $C_0 \leq 3.82 \cdot 10^{-10}$ and $9.1 \cdot 10^{-10} \leq C_0 < 11.6 \cdot 10^{-10}$, $\theta = 100$. This point agrees well with the soliton hierarchy first observed in[12].
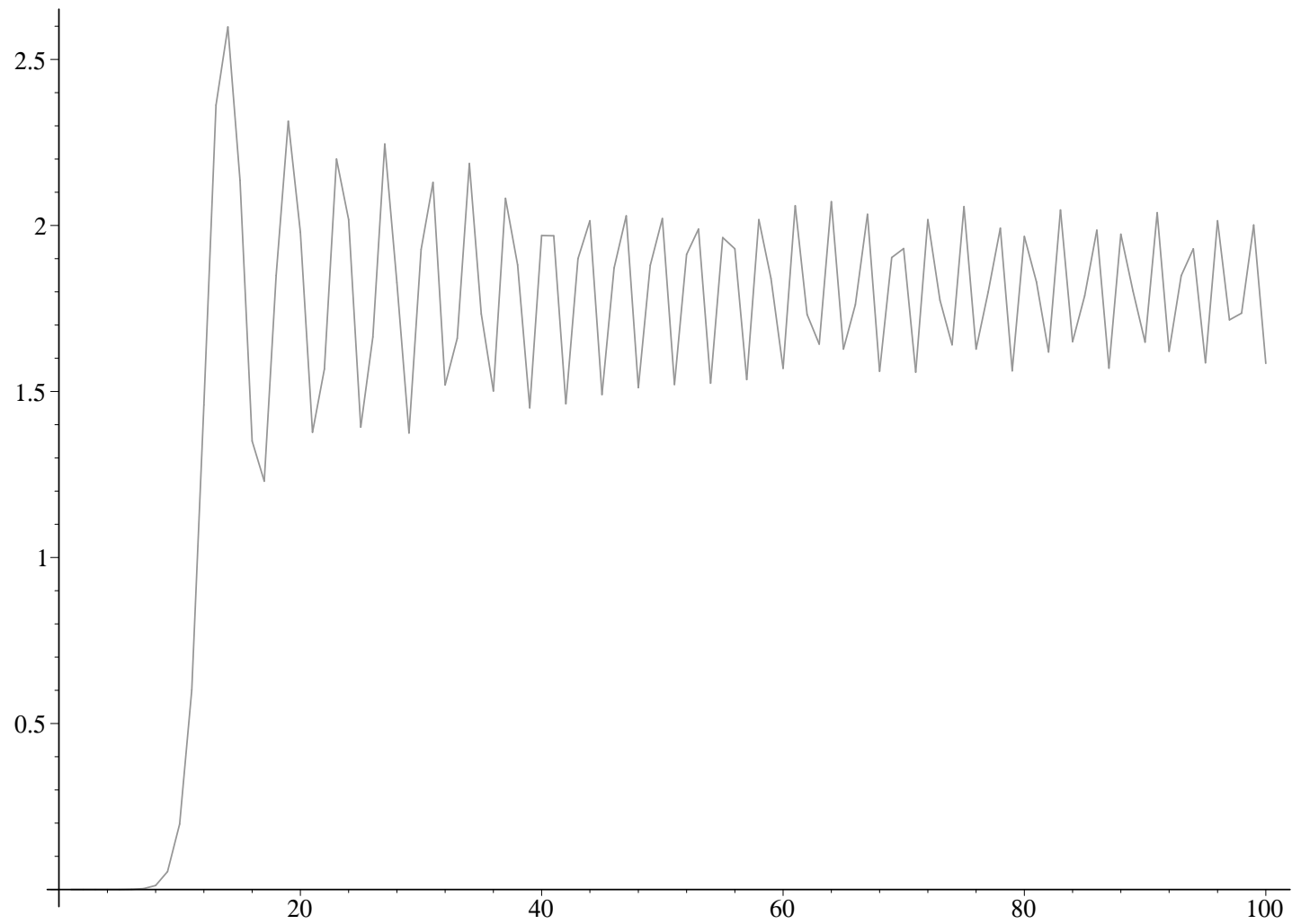
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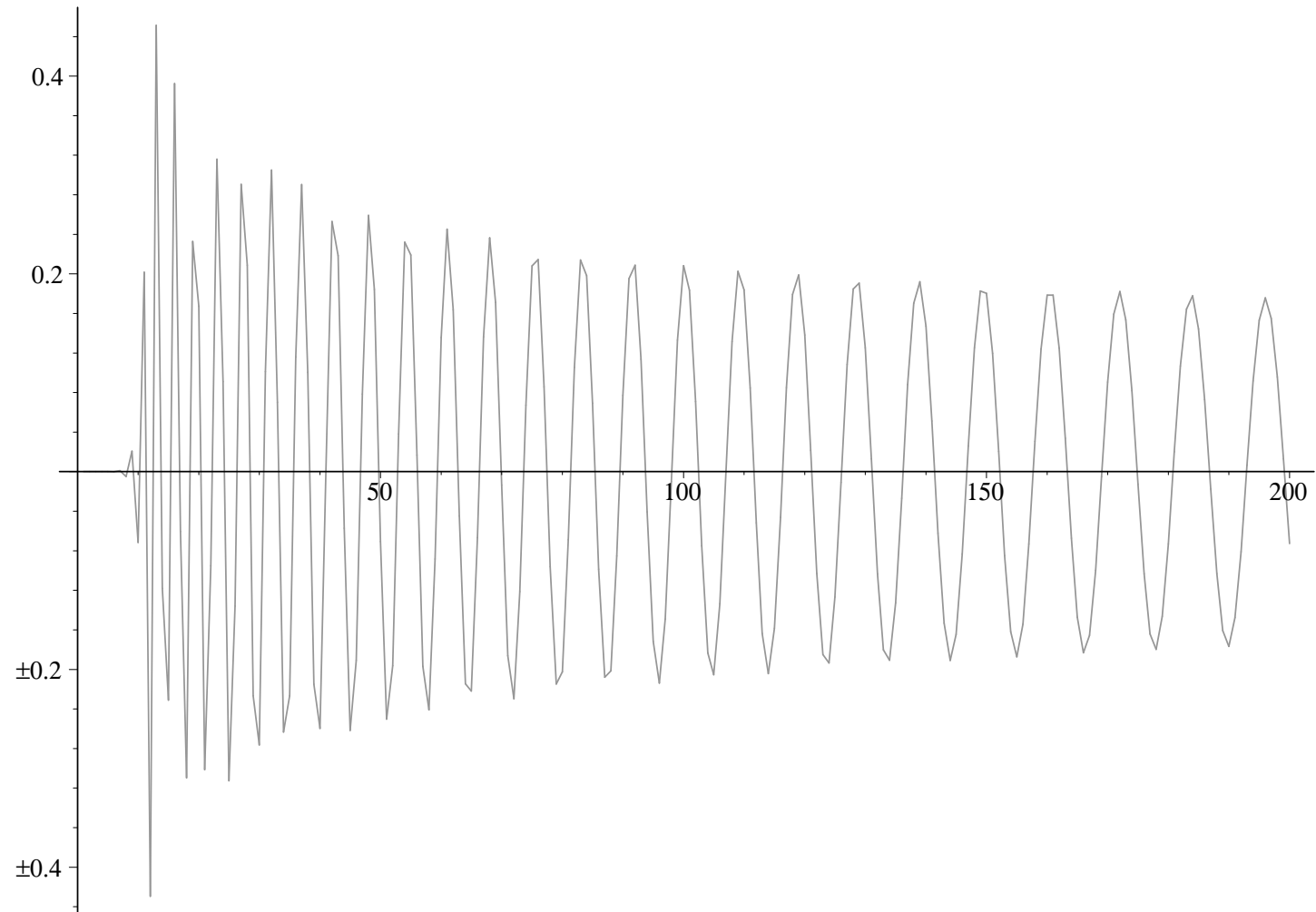
Dependence C(n)

Solution in configuration space: sum of the first 100 terms
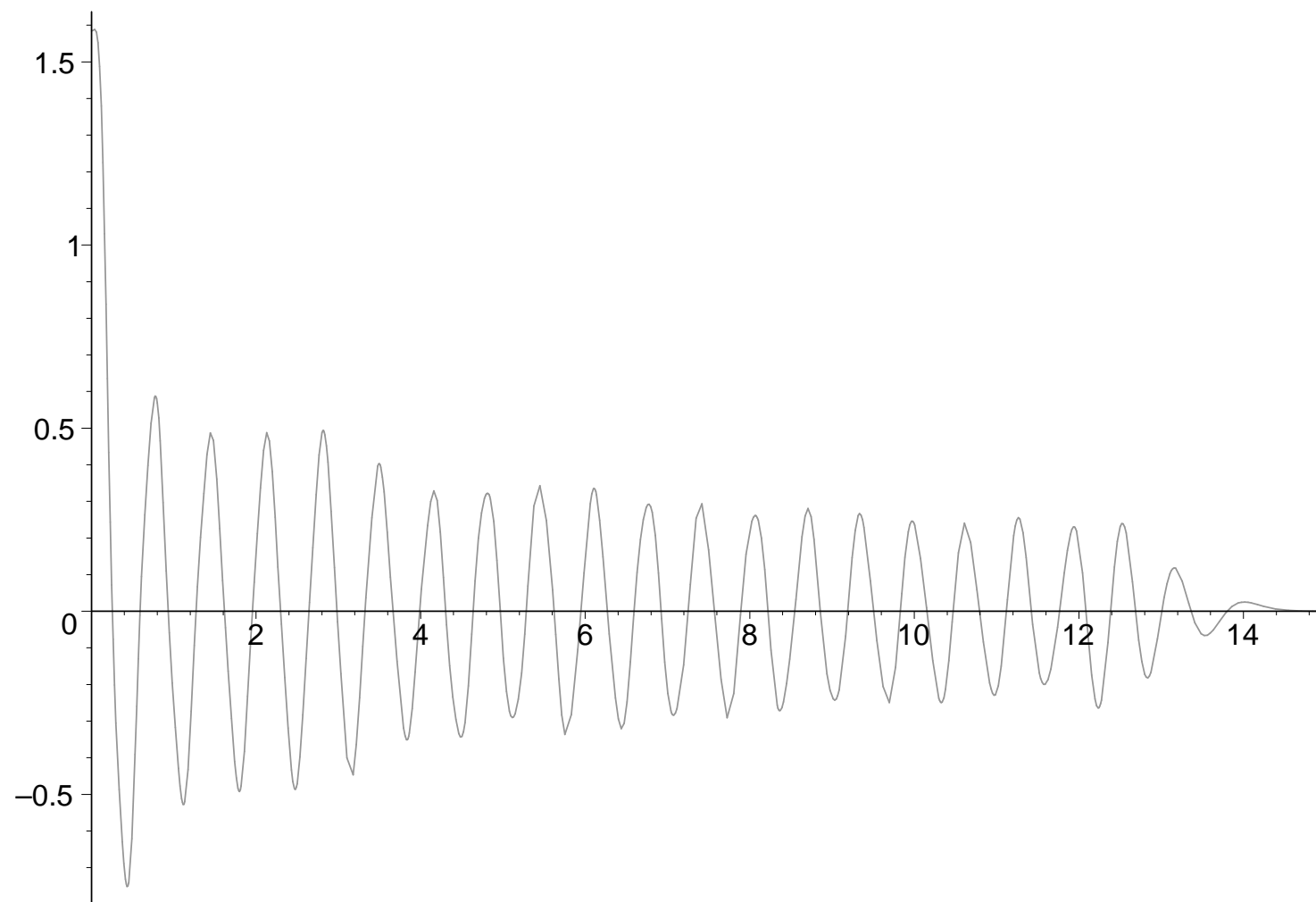
Dependence of maximum at the origin on number of terms in series

5   Concluding remarks

In the present paper we consider 2+1-dimensional noncommutative classical scalar field theory. The standard and basic tool for solving equations of motion at $\theta = \infty$ is the Weyl-Moyal correspondence between the product of operators acting on the Hilbert space and the nonlocal product of functions. We mention how one can solve the equations of motion in principle without referring to the Weyl-Moyal correspondence. Then the general ansatz for a radially symmetric solution is obtained confirming the celebrated results of[5]. Next these results are applied to studying generalized (distribution-valued) solutions and their physical assignment. The $\star$-product of distributions is defined as a distribution in the momentum space. Small excitations are shown to change the regularization. An approach to the accurate study of the radially symmetric solution dynamics is given and the effect of the time dynamics coming to the field rescaling is described. Then we proceed to examine the physics at finite θ. The effects of delocalization and taking down the degeneration are analyzed qualitatively and numerically. The special role of radially symmetric solutions is clarified.

There arise different questions of interest such as dynamics of fields without radial symmetry and general scattering problem. It is interesting to take quantum corrections into account. Among other problems there are coupling with the gauge field and searching out some closed form solution and analysis for finite θ.

6   Acknowledgements

7   Appendix

Derivation of the recurrent relations.   In the holomorphic representation the key role is played by the two operators

$$\hat{a} = \frac{\hat{q} - i\hat{p}}{\sqrt{2}},$$
$$\hat{a}^{\dagger} = \frac{\hat{q} + i\hat{p}}{\sqrt{2}}. \tag{48}$$

Commutation relations for them $[\hat{a}, \hat{a}^{\dagger}] = \hat{I}$ do have a representation in the space of analytic functions $f(\bar{z})$ with the inner product[13]

$$\langle f_1, f_2 \rangle = \int \frac{dz d\bar{z}}{2\pi 1} e^{(-\bar{z}z)} \overline{f_1(\bar{z})} f_2(\bar{z}). \tag{49}$$

The orthonormal state vectors are represented as

$$\mid n \rangle \leftrightarrow \psi_n(\bar{z}) = \frac{(\bar{z}^n)}{\sqrt{n!}}, \tag{50}$$

creation and annihilation operators as

$$\hat{a}^{\dagger} \leftrightarrow \bar{z},$$
$$\hat{a} \leftrightarrow \bar{\partial}. \tag{51}$$

An arbitrary operator $\hat{U}$ with matrix elements equal to

$$U_{mn} = \langle \psi_m \mid \hat{U} \mid \psi_n \rangle \tag{52}$$

can be represented as the integral operator with the kernel [9]

$$A(\bar{z}, z) = \sum_{m,n} U_{mn} \frac{\bar{z}^m}{\sqrt{m!}} \frac{z^n}{\sqrt{n!}}, \tag{53}$$

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[9] $A(z, \bar{w})$ is an analytic function (some series) of two complex arguments, and in general $z \neq w$.

$$(\hat{U}f)(\bar{z}) = \int \frac{dz\,d\bar{z}}{2\pi_1} e^{(-\bar{z}z)} A(\bar{z}, z) f(\bar{z}). \tag{54}$$

Now all the needed commutators in the equation of motion can be simply computed using the formulae [10]

$$\begin{aligned}
\hat{a}^\dagger \hat{U} &\leftrightarrow \bar{z} A(\bar{z}, z), \\
\hat{a}\hat{U} &\leftrightarrow \bar{\partial} A(\bar{z}, z), \\
\hat{U}\hat{a}^\dagger &\leftrightarrow \partial A(\bar{z}, z), \\
\hat{U}\hat{a} &\leftrightarrow z A(\bar{z}, z).
\end{aligned} \tag{55}$$

Under the assumption of the radial symmetry $U_{mn} = C_m \delta_{mn}$ the result is

$$(\partial\bar{\partial} - z\partial - \bar{z}\bar{\partial} + z\bar{z} - 1) A(\bar{z}, z) = \frac{\theta}{2} \sum_{n=0}^{\infty} \frac{\partial V}{\partial \phi}(C_n) \psi_n(\bar{z}) \overline{\psi_n(\bar{z})}, \tag{56}$$

and after the simple direct calculation of matrix elements one arrives to (39). The energy functional is obtained in a similar way.

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[10]The last two ones are obtained using integration by parts and reflect the fact that $\hat{a}$, $\hat{a}^\dagger$ are hermitian conjugate.